Brief comments on Jackiw-Teitelboim gravity coupled to Liouville theory

Gastón E. Giribet

Instituto de Astronomía y Física del Espacio, IAFE
C.C. 67 - Suc. 28, (1428) Buenos Aires, Argentina
and
Physics Department, University of Buenos Aires, FCEN-UBA
gaston@iafe.uba.ar

Abstract

The Jackiw-Teitelboim gravity with non-vanishing cosmological constant coupled to Liouville theory is considered as a non-critical string on d dimensional flat space-time. It is discussed how the presence of cosmological constant leads to consider additional constraints on the parameters of the theory, even though the conformal anomaly is independent of the cosmological constant. The constraints agree with the necessary conditions required to ensure that the tachyon field turns out to be a primary prelogarithmic operator within the context of the world-sheet conformal field theory. Thus, the linearized tachyon field equation allows to impose the diagonal condition for the interaction term.

We analyze the neutralization of the Liouville mode induced by the coupling to the Jackiw-Teitelboim lagrangian. The standard free field prescription leads to obtain explicit expressions for three-point functions for the case of vanishing cosmological constant in terms of a product of Shapiro-Virasoro integrals; this fact is a consequence of the mentioned neutralization effect.

1 Introduction

As it was early observed by Tseytlin, one of the principal motivations of the understanding of the dependence of the effective action on the tachyon field comes from the connection with studies of the two dimensional gravity [1]. Consequently, in this note we study the linearized tachyon field equation within the context of the description of the Jackiw-Teitelboim gravity [2] in terms of a non-linear σ-model describing the world-sheet dynamics of a non-critical string.

This class of analogies existing between two-dimensional models of gravitation and non-critical string theories was previously discussed in the literature. For instance, in reference [3], the action for two-dimensional gravity coupled to matter fields was quantized in the context of the conformal field theory method. In that reference, it was argued that the quantization could be carried out without restricting the dimension of the coupled matter. To do this, the author of [3] considered the case of vanishing cosmological constant $\Lambda = 0$ argueing that this case results suitable to take into account the general features of the model since the conformal anomaly is independent of the value of Λ. Regarding this assertion, in this brief note we show how additional restrictions really appear if a non-vanishing cosmological constant Λ is considered in the Jackiw-Teitelboim two dimensional theory of gravitation, even though the central charge of the model actually does not depend of the explicit value of Λ. This is achieved by the interpretation of the model as a consistent non-critical string theory.

In reference [4], the Jackiw-Teitelboim theory was also interpreted as a non-critical string theory in presence of background fields. Following this representation, we show that the additional restrictions on the coupling constant and the parameters of the model arise by solving the linearized tachyon field equation yielding from the mentioned interpretation of the model as a non-critical string theory.

The central point in the argumentation is based in the fact that the Jackiw-Teitelboim gravity coupled to Liouville theory and matter fields, as mentioned, could be geometrically interpreted as a conformal field theory describing the dynamic of a string on a d dimensional Minkowski target space-time. And, in the case of non-vanishing cosmological constant, the two dimensional world-sheet theory is conformed by sum of two parts: the first, a d-dimensional free field bosonic theory, and the second is an interaction term, like a screening charge, which in the general case is conformed by a logarithmic operator. Then, an additional constraint on the parameters of the theory is required in order to impose the necessary conditions to turn the interaction term into an integral of primary fields. This constraint, which is absent if $\Lambda = 0$ because it is automaticaly satisfied in such case, can be translated into a restriction on the functional form and value of the central charge of the theory.

It is important to mention that the coupling between two-dimensional models of gravity and Liouville theory was also studied in references [5, 6].

In the following paragraphs we study the model within the context of the conformal field theory formalism. We analyze the associated non-linear σ-model which can be recognized as the non-critical string theory on the d-dimensional

flat space-time coupled to linear dilaton background. Secondly, we explicitly show how the expression for three-point correlation functions in the case of vanishing cosmological constant can be written down in terms of Virasoro-Shapiro integrals because of the neutralization of Liouville mode induced by the constant curvature condition. This computation follows the steps of the standard Coulomb gas prescription.

2 Jackiw-Teitelboim gravity coupled to Liouville theory

Let us begin by considering the Jackiw-Teitelboim gravitational action coupled to Liouville theory and $d-2$ scalar fields representing the matter, namely

$$S_{eff} = S_{JT} + S_L + S_M + S_{ghost} \tag{1}$$

being

$$\begin{aligned}
S_{JT} &= \frac{b}{\pi}\int_M d^2x\sqrt{\hat{h}}\varphi\left(\tilde{R}+\tilde{\Lambda}\right) + \frac{\lambda}{4\pi}\int_M d^2x\sqrt{\hat{h}}\tilde{R} \\
S_L &= \frac{1}{\pi}\int_M d^2x\sqrt{h}\left(ah^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma + QR\sigma + \frac{\mu}{2}e^{\gamma\sigma}\right) \\
S_M &= \frac{1}{4\pi}\int_M d^2x\sqrt{h}h^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j\delta_{ij}
\end{aligned}$$

where a and b are (positive) real parameters interpreted as coupling constant of the Jackiw-Teitelboim lagrangian and the Liouville lagrangian respectively; $\tilde{\Lambda}$ represents the target space cosmological constant, while μ is the quoted Liouville cosmological constant; on his turn, R refers to the Ricci scalar corresponding to the two-dimensional metric $h_{\alpha\beta}$ on the two-manifold M, while $\tilde{R}$ corresponds to the Ricci scalar constructed from the rescaled metric $\hat{h}_{\alpha\beta}$ [7], defined by

$$\hat{h}_{\alpha\beta} = e^{\gamma\sigma}h_{\alpha\beta} \tag{2}$$

where $(\alpha,\beta)\in\{0,1\}$.

The field φ is the Jackiw-Teitelboim scalar field, while σ is the Liouville scalar field; there exist other $d-2$ scalar fields which represent the matter, being denoted by X^i, $i\in\{2,3...d-1\}$. The Hilbert term also appears in the action (1) contributing with the Euler characteristic number $\chi(h)$; and where λ is a real coupling constant which can be related with the dilaton zero-mode. On the other hand, the ghost action S_{ghost} needs to be included; it can be expressed in terms of the b-c system as

$$S_{ghost} = \frac{1}{2\pi}\int_M d^2x\sqrt{h}h^{\alpha\beta}c^\rho\nabla_\alpha b_{\beta\rho} \tag{3}$$

Notice that the Jackiw-Teitelboim field φ acts as a Lagrange multiplier, which can be integrated out introducing the constant curvature constraint

$$\tilde{R}+\tilde{\Lambda}=0.$$

As introduced before, this class of two-dimensional gravitational action coupled to Liouville theory was treated in different contexts. Indeed, an interesting analysis of its perturbative aspects was effectuated in [4, 8]; in reference [3] this action was quantized by using the path integral approach and conformal field methods too; it was also studied in the context of W_3 gravity [9], and a careful analysis of its classical and quantum features was presented in [10]. In [3, 11] it was shown that the Teitelboim-Jackiw lagrangian appears as the metric version of a topological construction based on the $SO(1,2)$ group. This realization relies in the philosophy of treating the two-dimensional models of gravity as a kind of projective reduction of three-dimensional Chern-Simon theory. See also [12] for an interesting discussion.

Here, we shall consider the classical model defined by action (1) as the starting point; with this purpose, let us define the following variables

$$\sigma = \frac{\kappa}{\gamma}\left(X^1 + X^0\right) \tag{4}$$

$$\varphi = \frac{\kappa}{\gamma}\left(X^1 + \xi X^0\right) \tag{5}$$

where $\kappa = \frac{\gamma}{2\sqrt{a+\gamma b}}$ and $\xi = -1 - \frac{2a}{\gamma b}$. These variables are well defined in terms of the original coordinates φ and σ iif $a + b\gamma \neq 0$. In terms of the new coordinates, the above action can be written as follows

$$S_{eff} = S_0 + S_I + \lambda\chi(h) \tag{6}$$

being

$$S_0 = \frac{1}{4\pi}\int d^2z\left(\eta_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu + RQ_0X^0 + RQ_1X^1\right), \tag{7}$$

$$S_I = \frac{1}{4\pi}\int d^2z\left(\xi\Lambda X^0 + \Lambda X^1 + 2\mu\right) e^{\kappa(X^0+X^1)} \tag{8}$$

Here, $\eta_{\mu\nu}$ denotes the d-dimensional Minkowski metric and the following notation was introduced

$$Q_0 = 2\frac{Q - b - 2a/\gamma}{\sqrt{a+\gamma b}}$$

$$Q_1 = 2\frac{Q + b}{\sqrt{a+\gamma b}}$$

In (7) and (8) z and $\bar{z}$ denote the usual world-sheet complex coordinates, being $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$. Notice that the cosmological constant Λ was also rescaled by a constant factor, namely $\Lambda = \frac{2b}{\sqrt{a+\gamma b}}\tilde{\Lambda}$. We are considering greek indices which cover the d-dimensional space-time, $(\mu,\nu) \in \{0, 1, ...d-1\}$.

Indeed, it would be also interesting to note that the constant curvature constraint can be derived by combining the Euler-Lagrange equation for X^0 and X^1 fields, obtaining

$$e^{-\gamma\sigma}\left(R - \gamma\partial^2\sigma\right) + \tilde{\Lambda} = 0,$$

which can be written in the usual form $\tilde{R} + \tilde{\Lambda} = 0$ if the relation (2) is taken into account.

Thus, we have written the two-dimensional action (1) in terms of a theory of auto-interacting scalar fields coupled to classical gravity in the conformal gauge (2); it will be a useful picture to work out the features of the model.

3 The conformal field theory

The above free action S_0 defines a two-dimensional conformal quantum field theory, which could be analyzed by the usual techniques of 2D CFT. Indeed, the holomorphic component of the stress-tensor corresponding to this model is given by

$$T(z) = -\frac{1}{2}\eta_{\mu\nu}\partial X^\mu \partial X^\nu + \frac{Q_0}{2}\partial^2 X^0 + \frac{Q_1}{2}\partial^2 X^1 \tag{9}$$

with free field bosonic correlators,

$$\langle X^\mu(z) X^\nu(w)\rangle = -\eta^{\mu\nu}\ln(z-w) \tag{10}$$

Thus, it is easy to show that the following operator product expansion holds

$$T(z)T(w) = \frac{\frac{d}{2} + \frac{3}{2}(Q_1^2 - Q_0^2)}{(z-w)^4} + \frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + ...$$

From this expansion, we obtain that the central charge of the model is given by

$$c = d + 48\left(\frac{Q}{\gamma} - \frac{a}{\gamma^2}\right) \tag{11}$$

which does not include the ghost contribution $c_{ghost} = -26$.

On the other hand, the interaction term S_I is not an integral of primary fields as it occurs in the usual treatement of two-dimensional models with conformal invariance. In fact, the operator

$$\mathcal{T}_\Lambda(z) = \left(\xi\Lambda X^0(z) + \Lambda X^1(z) + 2\mu\right) e^{\kappa X^0(z) + \kappa X^1(z)} \tag{12}$$

turns out to be a logarithmic operator as it can be shown by the emergence of non-diagonal terms in the operator product expansion with the stress-tensor of the free field theory, namely

$$T(z)\mathcal{T}_\Lambda(w) = h\frac{\mathcal{T}_\Lambda(w)}{(z-w)^2} + \varepsilon\frac{\mathcal{T}_{\Lambda=0}(w)}{(z-w)^2} + \frac{\partial\mathcal{T}_\Lambda(w)}{(z-w)} + ... \tag{13}$$

$$T(z)\mathcal{T}_{\Lambda=0}(w) = h\frac{\mathcal{T}_{\Lambda=0}(w)}{(z-w)^2} + \frac{\partial\mathcal{T}_{\Lambda=0}(w)}{(z-w)} + ... \tag{14}$$

being

$$\mathcal{T}_{\Lambda=0}(z) = 2\mu e^{\kappa(X^0 + X^1)}$$

which is actually a primary field with conformal dimension h; being

$$h = -\frac{\kappa}{2}\left(Q_0 - Q_1\right) \tag{15}$$

$$\varepsilon = \frac{\Lambda}{2\mu}\left(\kappa\left(\xi - 1\right) - \frac{\xi}{2}Q_0 + \frac{1}{2}Q_1\right) \tag{16}$$

Then, in order to realize the theory as a 2D CFT representing a non-critical string theory, we have to impose the condition $h = 1$, which is directly satisfied by the original definition of the Liouville cosmological term, being proportional to $e^{\kappa(X^0+X^1)}$.

An question could appear in this point: it could be possible to ask whether the constraints $h = 1$ and $\varepsilon = 0$ represent additional restrictions which need to be imposed on the theory indeed; in fact, these constraints are necessary conditions required by the stringy interpretation of the model as a consistent non-critical theory. The point is that, since our further analysis involving the β-function equations will permit to recover the equations (15) and (16) as necessary conditions for conformal invariance of the model, we see these equations as the necessary constraints to be impossed in order to obtain a quantum conformal field theory. In the next section, we will remark this point with more detail.

On the other hand, in order to make the charge $\mathcal{T}_\Lambda(w)$ a primary field, we also need to impose the constraint $\varepsilon = 0$ which turns the operator (12) into a diagonal (prelogarithmic) operator. Thus, we obtain the following condition

$$2\Lambda\kappa\left(\xi - 1\right) = \Lambda\left(\xi Q_0 - Q_1\right) \tag{17}$$

which can be written in terms of the original parameters as follows

$$\tilde{\Lambda}\left(a + b\gamma\right)\left(\gamma^2 - 2Q\gamma + 4a\right) = 0 \tag{18}$$

This equation, moreover the trivial case $\Lambda = 0$, has solutions of the form

$$\gamma = Q \pm \sqrt{Q^2 - 4a} \tag{19}$$

This is an additional restriction on the value of central charge, which would permit to realize the interaction term in the context of the CFT method. Note that $\xi = 1$ also solves the equation (18); however, it is not an allowed value since in such case the transformation (4)-(5) becomes singular and the equivalence between the original gravity action and the linear dilaton background (6) breakes down.

The introduction of the contraint (19) is not a minor detail; notice that if it is replaced in (11) the central charge of the model becomes

$$c = d + 24 + \frac{48a}{\gamma^2} \tag{20}$$

which can not be set to 26 for the case $d \geq 2$, $a > 0$. On the other hand, this constraint excludes particular cases like $Q = a$, $\gamma = 2$, which were considered

in the $\Lambda = 0$ case as examples of a critical string theory [4]. On his turn, the $Q = \frac{3}{2}$ model receives particular interest in the context of studies of W_3 gravity [9]. Notice that condition (19) is not present in the $\Lambda = 0$ case since the interaction term becomes primary in this point. Of course, (19) can be obtained directly by using the original variables, for instance by solving the linearized tachyon equation in the case of non-vanishing Λ in the coordinates adopted in reference [4]. On his turn, in terms of the nomenclature of reference [3] this condition can not be satisfied because of the particular fixation of the value of the background charge Q.

Then, we are compeled to state that an additional condition impossed on the central charge arises if cosmological constant is included in the Jackiw-Teitelboim model coupled to Liouville theory. And, this new restriction is independet of the explicit value of Λ.

This proposal needs to be contrasted with the *healing* proposed in reference [11], where, instead imposing condition (19), it was suggested to change the tachyon field by adding a new term proportional to $\sigma e^{\gamma\sigma}$, which could appear in certain unknown step in the renormalization procedure applied to the non-local Polyakov action.

It would be important to mention that similar operators $\sim \varphi e^{\gamma\sigma}$ appear in other treatements of two-dimensional gravity; for instance, see [13].

4 Linearized tachyon field equation

On the other hand, we can analyze the above restriction on the central charge within the context of the β-function equations for this model. In fact, as it was observed early [3, 4], the whole action $S_0 + S_I$ can be interpreted as a world-sheet string action formulated on Minkowski target space-time in presence of non-trivial background fields $\Phi(X)$ (the dilaton) and $\mathcal{T}(X)$ (the tachyon), whose classical configurations are given by

$$\Phi(X) = \frac{\lambda}{2} + \frac{1}{2} Q_\mu X^\mu \tag{21}$$

$$\mathcal{T}_\Lambda(X) = \left(\Lambda \zeta_\mu X^\mu + 2\mu\right) e^{\kappa_\nu X^\nu} \tag{22}$$

Notice that the functional form (22) is not a particular case within the context of the theory of gravity defined by action S_{eff}; moreover, this form is a more general *ansatz* which includes the particular case of the tachyonic configuration arising in (8).

The β-function equations $\beta^\Phi = \beta^G_{\mu\nu} = \beta^B_{\mu\nu} = 0$ are immediately satisfied at the lowest order in α' by the condition $c = 26$. However, the linearized tachyon field equation $\beta^{\mathcal{T}} = 0$ needs to be imposed by means of additional functional relations between the parameters, namely

$$\beta^{\mathcal{T}} = -\frac{1}{2} \nabla_\mu \nabla^\mu \mathcal{T}_\Lambda + \nabla_\mu \Phi \nabla^\mu \mathcal{T}_\Lambda - \frac{2}{\alpha'} \mathcal{T}_\Lambda + ... = 0 \tag{23}$$

where we neglected highest orders in the α' expansion and, then, we can adopt the convention $\alpha' = 2$ (note that our convention will be consequent with $\alpha' = 2$. For instance, see eqs. (2.5.1a), (3.7.6) and (9.9.2) of reference [14]. Similarly to the convention adopted in [4], our notation is congruent with the one used in [1]).

The configuration (22) represents a small tachyon, since we are considering perturbation theory in α'. The leading terms in the tachyon effective action are linear in $\mathcal{T}_\Lambda$; neglecting the higher powers of tachyonic field is consequent with dealing with this field as it would not be perturbating the background as a source of geometry.

Then, it is necessary to solve the linearized equation

$$-\partial_\mu \partial^\mu \mathcal{T}_\Lambda + Q_\mu \partial^\mu \mathcal{T}_\Lambda - 2\mathcal{T}_\Lambda = 0 \tag{24}$$

which admits a solution of the form (22) if the following constraints hold

$$\kappa_\mu (Q^\mu - \kappa^\mu) = 2 \tag{25}$$

$$\zeta_\mu (Q^\mu - 2\kappa^\mu) = 0 \tag{26}$$

Clearly, from (25) we can see that in the absence of the linear dilaton field ($Q_\mu = 0$) the mass-shell condition for the tachyon field $\kappa_\mu \kappa^\mu = -2$ is recovered. Here, we have $c_{gravity} = 2 + 3Q_\mu Q^\mu$.

In our case, we are dealing with the particular configuration given by

$$\begin{aligned} \zeta_\mu \zeta^\mu &= \Lambda^2 \frac{4a}{b\gamma}\left(\frac{4a}{b\gamma} + 1\right) \\ Q_\mu Q^\mu &= \frac{16}{\gamma}\left(Q - \frac{a}{\gamma}\right) \\ \kappa_\mu \kappa^\mu &= 0 \end{aligned}$$

i.e. in the coordinates defined above we have $\zeta_\mu = \Lambda(\xi, 1, 0, 0...0)$, $Q_\mu = (Q_0, Q_1, 0, 0, ...0)$ and $\kappa^\mu = (-\kappa, \kappa, 0, 0, ...0)$. Then we can note that, in terms of the original parameters, the constraint (26) demands

$$\tilde{\Lambda}\,(a + \gamma b)\left(\gamma^2 - 2Q\gamma + 4a\right) = 0, \tag{27}$$

which exactly agrees with the condition (18), which we obtained above by considerations about the prelogarithmic condition $\varepsilon = 0$ for the interaction term in the CFT action. Of course, the agreement between (27) and (18) has not to be a surprise since the linearized tachyon field equation manifestly represents the requirements of conformal invariance for the interaction term [1, 16, 17].

Thus, we find that the β-function equation for the tachyon field, $\beta^{\mathcal{T}} = 0$, allows to obtain the prelogarithmic condition $\varepsilon = 0$ for the operator $\mathcal{T}_\Lambda$ in the world-sheet conformal field theory. Moreover, the logarithmic part of the operator product expansion $T(z)\mathcal{T}_\Lambda(w)$ is parametrized by the values (Q, γ), and these diagonalize the L_0 operator in the critical point satisfying (19). Notice

that this argumentation is independent of the explicit value of the non-vanishing cosmological constant Λ.

Hence, a new restriction would appear if cosmological constant is considered. In the general case, it is not possible to interpret the theory as a critical string theory as it was done in the $\Lambda = 0$ case, which would appear as a disctintive particular case.

As in the case of the exponential functional form of $\mathcal{T}_{\Lambda=0}$, the above solution for the tachyon field in the linear dilaton background solves the lowest orders in α' of β-function equations only for the lowest order in powers of the field $\mathcal{T}_{\Lambda}$.

The solution (22) is a simple generalization of the David-Distler-Kawai background [7, 15]. From (11), (25) and (26), we can write

$$\kappa_\mu \kappa^\mu - \kappa_\mu Q^\mu + \frac{1}{4} Q_\mu Q^\mu + \frac{d-2}{12} = 0 \tag{28}$$

With the intention to analyze the general aspects of the solutions of such equation of motion we can see that in the case of κ^μ being a time-like vector of the particular form $\kappa^\mu = (\kappa, 0, 0, ...0)$, the solution for the linearized tachyon field takes the form $\kappa^\mu = -\frac{1}{2}\left(Q_0 \pm \sqrt{Q_1^2 + \frac{d-2}{6}}\right)\delta^{0\mu}$, whose qualitative behaviour depends on the relative value of d and $Q_{1.}$ On his turn, in the case of κ^μ being a space-like vector, $v.g.$ the case $\kappa^\mu = (0, \kappa, 0, ...0)$, we have $\kappa^\mu = \frac{1}{2}\left(Q_1 \pm \sqrt{Q_0^2 - \frac{d-2}{6}}\right)\delta^{1\mu}$. The physical aspects of such solutions were extensively studied within the context of non-critical string theory where it was discussed how the tachyon field acts as a barrier of potential competeing with the action of the linear dilaton background, see for instance [7, 14, 15, 16, 17].

Moreover, the case we are interested on is the analogous to the Jackiw-Teitelboim model coupled to Liouville theory, which corresponds to the light-like case $\kappa_\mu \kappa^\mu = 0$. In this case, the solutions fall into the region described by the constraint

$$\kappa_\mu Q^\mu = 2 \tag{29}$$

which is precisely the mass shell condition $h = 1$ for the tachyon field in this dilatonic configuration. Here, the null condition $\kappa_\mu \kappa^\mu = 0$ appears as the manifestation of the neutralization effect of Liouville mode induced by the constant curvature constraint, which was studied in reference [3] within the context of the path integral approach. The mentioned light-like condition (29) can be seen as a change in the propagator of Liouville mode [3, 4, 7] and it leads to the linear functional form for the conformal dimension of the operators with the form $\sim e^{\kappa(X^0+X^1)}$, instead the quadratic dependence which leads to the definition of two-diferent screening charges, Weyl-like reflection relations and the existence of conjugated pictures in the case of similar conformal models. Let us postpone some comments about this for the next section.

Here, the intention here was to explore the general features of including a non-vanishing cosmological constant Λ in the context of the stringy interpretation of the Jackiw-Teitelboim gravity coupled to Liouville theory.

5 Three-point correlation functions

Now, by using this CFT description of the model, it is feasible to follow the steps of the Coulomb gas prescription with the purpose to write down explicit expressions for the integral representation of correlation functions in the theory defined by the action (6). For instance, in reference [3] it was pointed out the importance of extending the CFT analysis with the purpose to represent correlation functions in this model; on the other hand, in [21], a similar treatement of three-point functions was presented for the case of minimal models coupled to two-dimensional gravity. Indeed, it is possible to compute the three-point functions in the Jackiw-Teitelboim gravity coupled to Liouville theory for the case $\Lambda = 0$ by using the quoted free field realization. In fact, the N-point correlation functions of primary fields $\Phi_k(z) = \exp ik_\mu X^\mu(z)$ on the sphere are described by the following form

$$\left\langle \prod_{i=1}^{N} \Phi_{\alpha^{(i)},\beta^{(i)},k^{(i)}}(z_i) \right\rangle_{S_{eff}} = e^{-2\lambda}\mu^t \Gamma(-t) \prod_{r=1}^{t} \int d^2 w_r \left\langle \prod_{i=1}^{N} \Phi_{\alpha^{(i)},\beta^{(i)},k^{(i)}}(z_i) \prod_{r=1}^{t} \mathcal{T}_{\Lambda=0}(w_r) \right\rangle_{S_0}$$

where standard formulae of the path intergal realization is involved (see, for instance [16, 18, 19, 20, 21]) and we have defined $\alpha = ik_0$, $\beta = ik_1$. By using the projective invariance, three different inserting points of vertex operators can be fixed on the world-sheet in order to cancellate the volume of the $SL(2,C)$ group, $v.g.$ being $(z_1, z_2, z_3) = (0, 1, \infty)$. Thus, in the usual way, we can write the N-point correlation functions $G^N(z_1, z_2, ...z_N)$ as

$$G^N_{k^{(1)}_{\mu_1}, k^{(2)}_{\mu_2}, ...k^{(N)}_{\mu_N}}(z_1, z_2, ...z_N) = \left\langle \prod_{i=1}^{N} \Phi_{\alpha^{(i)},\beta^{(i)},k^{(i)}}(z_i) \right\rangle_{S_{eff}} =$$

$$\begin{aligned} = \quad & \frac{(2\pi)^{d-1}(2\mu)^t e^{-2\lambda}}{\kappa} \Gamma(-t) \prod_{i>j}^{N,N-1} |z_i - z_j|^{2\left(k^{(i)}k^{(j)} + \alpha^{(i)}\alpha^{(j)} - \beta^{(i)}\beta^{(j)}\right)} \times \\ & \times \prod_{r=1}^{t} \int d^2 w_r \prod_{r=1}^{t}\prod_{i=1}^{N} |z_i - w_r|^{2\kappa(\alpha^{(i)} - \beta^{(i)})} \times \\ & \delta\left(\sum_{i=1}^{N} (\alpha^{(i)} - \beta^{(i)}) + Q_0 - Q_1 \right) \delta^{(d-2)}\left(\sum_{i=1}^{N} k^{(i)} \right) \end{aligned}$$

where we denoted $t = \frac{1}{2\kappa}\left(Q_0 + Q_1 - \sum_{i=1}^{N}(\alpha^{(i)} + \beta^{(i)})\right)$. The delta functions and the global factor $\frac{(2\pi)^{d-1}(2\mu)^t}{\kappa}$ yield from the integration over the zero-modes of X^μ fields, while the factor $e^{-2\lambda}$ stands by the Einstein-Hilbert term in the genus zero contribution of the functional measure $e^{-S_{eff}}$. In order to make the correlation functions non-vanishing we get $\sum_{i=1}^{N}\left(\alpha^{(i)} - \beta^{(i)}\right) = Q_0 - Q_1$. Thus,

we can write the conservation laws as

$$\sum_{i=1}^{N} \alpha^{(i)} \pm \sum_{i=1}^{N} \beta^{(i)} = (Q_0 - t\kappa) \pm (Q_1 - t\kappa) \tag{30}$$

with t being a non-negative integer.

Notice that mixed terms of the form $|w_r - w_{r'}|^{2\delta}$ do not exist because of the particular configuration $\delta = -\kappa_\mu \kappa^\mu = 0$. This fact makes that the integration over the different screening insertions turns out to be simplified leading to obtain a direct product of separated integrals. This is a manifestation of the neutralization effect of the Liouville mode pointed out before; *i.e.* in the case of three-point functions, the Dotsenko-Fateev type integrals [18, 19, 20, 21] become a product of Virasoro-Shapiro integrals because of the condition $\kappa_\mu \kappa^\mu = 0$.

Then, we can explicitly compute the three-point correlators as follows

$$G^3_{k^{(1)}_{\mu_1}, k^{(2)}_{\mu_2}, k^{(3)}_{\mu_3}}(0,1,\infty) = \left\langle \Phi_{\alpha^{(1)},\beta^{(1)},k^{(1)}}(0) \Phi_{\alpha^{(2)},\beta^{(2)},k^{(2)}}(1) \Phi_{\alpha^{(3)},\beta^{(3)},k^{(3)}}(\infty) \right\rangle =$$

$$\begin{aligned} = \quad & \frac{(2\pi)^{d-1} (2\mu)^t e^{-2\lambda}}{\kappa} \Gamma(-t) \prod_{r=1}^{t} \int d^2 w_r \prod_{r=1}^{t} |w_r|^{2\kappa(\alpha^{(1)}-\beta^{(1)})} |1-w_r|^{2\kappa(\alpha^{(2)}-\beta^{(2)})} \times \\ & \times \delta\left(\sum_{i=1}^{3} (\alpha^{(i)} - \beta^{(i)}) + Q_0 - Q_1 \right) \delta^{(d-2)} \left(\sum_{i=1}^{3} k^{(i)} \right) \end{aligned}$$

obtaining

$$\begin{aligned} G^3_{k^{(1)}_{\mu_1}, k^{(2)}_{\mu_2}, k^{(3)}_{\mu_3}} = & \frac{(2\pi)^{d-1} e^{-2\lambda}}{\kappa} \Gamma\left(\frac{1}{\kappa} \left(\alpha^{(1)} + \alpha^{(2)} + \alpha^{(3)} - Q_0 \right) \right) \times \\ \times & \left(2\pi\mu \frac{\gamma(1+\kappa(\alpha^{(1)}-\beta^{(1)}))\gamma(1+\kappa(\alpha^{(2)}-\beta^{(2)}))}{\gamma(2+\kappa(\alpha^{(1)}+\alpha^{(2)}-\beta^{(1)}-\beta^{(2)}))} \right)^{\frac{1}{\kappa}\left(Q_0 - \sum_{i=1}^{3} \alpha^{(i)}\right)} \end{aligned}$$

where $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$. By considering the condition $Q_1 - Q_0 = \frac{2}{\kappa}$ and noting that $\gamma^{-1}(x) = \gamma(1-x)$, we can write $\prod_{i=1}^{3} \gamma^{-1}(\kappa(\beta^{(i)} - \alpha^{(i)})) = \frac{1}{2} \prod_{i \neq j} \gamma(\kappa(\beta^{(i)} + \beta^{(j)} - \alpha^{(i)} - \alpha^{(j)}) - 1)$; then, the above expression can be rewritten as

$$G^3_{k^{(1)}_{\mu_1}, k^{(2)}_{\mu_2}, k^{(3)}_{\mu_3}} = \frac{(2\pi)^{d-1} e^{-2\lambda}}{\kappa} \Gamma\left(\sum_{i=1}^{3} \frac{\alpha^{(i)}}{\kappa} - \frac{Q_0}{\kappa} \right) \left(2\pi\mu \prod_{i=1}^{3} \frac{\Gamma\left(1+\kappa(\alpha^{(i)}-\beta^{(i)})\right)}{\Gamma\left(\kappa(\beta^{(i)}-\alpha^{(i)})\right)} \right)^{\frac{1}{\kappa}\left(Q_0 - \sum_{i=1}^{3} \alpha^{(i)}\right)} \tag{31}$$

satisfying (30). These would be the general form for three-point functions if the standard Coulomb gas prescription is adopted for computing the correlators.

Let us make a last comment about the role of φ field. Indeed, it is usual to consider an additional dynamical term for φ (see for instance [4]), namely

$$S_\varphi = \frac{g}{4\pi} \int d^2 z \sqrt{h} h^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi \tag{32}$$

Notice that if such a term is included, the case $\xi \to \xi = -\frac{2\gamma a + b}{2g + b}$ needs to be considered in (5) in order to diagonalize the metric of target space. In this case, we would obtain a non-vanishing correlator $\langle \sigma(z)\sigma(w) \rangle$ for the Liouville mode (*i.e.* $\kappa_\mu \kappa^\mu \neq 0$). Then, the inclusion of a dynamical term for φ field restores the influence of the σ field, which became harmless because of the presence of the constant curvature constraint. On his turn, if (32) is included, the three-point correlators are given in terms of Dotsenko-Fateev type integrals $\prod_{r=1}^{t} \int d^2 w_r \prod_{r=1}^{t} |w_r|^{2(\kappa_0 \alpha^{(1)} - \kappa_1 \beta^{(1)})} |1 - w_r|^{2(\kappa_0 \alpha^{(2)} - \kappa_1 \beta^{(2)})} \prod_{r'<r}^{t-1,t} |w_{r'} - w_r|^{2(\kappa_0^2 - \kappa_1^2)}$ which can be performed by using the result of reference [20] (specificaly, see eq. (B.9) of this paper). In such case, a global factor of the form $\Gamma(t+1) \prod_{r=0}^{t} \frac{\gamma(r(\kappa_0^2 - \kappa_1^2)/2)}{\gamma((\kappa_0^2 - \kappa_1^2)/2)}$ stands in the expression for three-point correlators; it is easy to see that in the $\kappa_\mu \kappa^\mu \to 0$ limit, where the Liouville mode becomes harmless, the expression (31) is recovered.

6 Conclusions

The implications of considering a non-vanishing cosmological constant in Jackiw-Teitelboim two-dimensional theory of gravity coupled to Liouville theory have been discussed. This was achieved by the treatement of the model as a non-critical string theory. Within the context of this stringy interpretation, we emphasize that the presence of a non-zero value for Λ demands a non trivial treatement of the coupling of both actions. Indeed, additional restrictions on the value of the central charge of the model need to be considered even though the conformal anomaly is independent of the explicit value of Λ. The constraint, which realize the additional restriction, agrees with the condition necessary to turn the tachyon field operator $\mathcal{T}_\Lambda(z)$ into a primary (prelogarithmic) operator [22] in the context of the world-sheet conformal field theory. As mentioned, the linearized tachyon field equation leads to impose the diagonal condition for the operator product expansion $T(z)\mathcal{T}_\Lambda(w)$.

We have seen how the coupling between two-dimensional gravity and Liouville theory leads to the neutralization of the Liouville mode as it was pointed out in reference [3], where it was explained how the constant curvature constraint renders the Liouville action *harmless* by *trivializating the σ dependence.* This neutralization effect is reflected, within the context of the conformal field theory description, by means of the null condition $\kappa_\mu \kappa^\mu = 0$, which would permit to realize the correlation functions in terms of a direct product of multiple integrals in the whole complex plane. Then, explicit expression for three-point functions were obtained in terms of Virasoro-Shapiro integrals which yield from the neutralization condition imposed on the Dotsenko-Fateev type realization.

These results have been obtained by means of the description of the model in the context of the 2D CFT formalism, which, once again, turns out to be a fruitful method to study several features of two-dimensional quantum gravity.

I wish to thank Noureddine Mohammedi and Mariano Galvagno. This work was supported by Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) and University of Buenos Aires (UBA).

References

[1] A.A. Tseytlin, Phys.Lett. **B264**, 311 (1991)

[2] C. Teitelboim, Phys.Lett. **B126**, 41 (1983). R. Jackiw, in Quantum Theory of Gravity, ed. S. Christiansen (Hilger, 1984). C. Teitelboim, in Quantum Theory of Gravity, ed. S. Christiansen (Hilger, 1984)

[3] A.H. Chamseddine, Phys.Lett. **B256**, 379 (1991)

[4] D.F. Mazzitelli and N. Mohammedi, Nucl.Phys. **B401** (1993) 239. arXiv:hep-th/9109016

[5] S.D. Odintsov and I.L. Shapiro, Phys.Lett. **B263**, 183 (1991). S.D. Odintsov and I.L. Shapiro, Europhys.Lett. **15**, 575 (1991). S.D. Odintsov and I.L. Shapiro, Int.J.Mod.Phys. **D1**, 571 (1993). S.Nojiri and S.D. Odintsov, Int.J.Mod.Phys. **A16**, 1015 (2001). E.Elizalde, S. Naftulin and S.D. Odintsov, Int.J.Mod.Phys. **A9**, 933 (1994)

[6] R.B. Mann, Nucl. Phys. **B418** (1994) 231. J. Kamnitzer and R.B. Mann, Nucl.Phys. **B609** (2001) 429-441

[7] F. David, Mod.Phys.Lett. **A3** (1988) 1651. A.M. Polyakov, Phys.Lett. **B163** (1981) 207. A.M. Polyakov, Mod.Phys.Lett. **A2**, 899 (1987)

[8] D.F. Mazzitelli and N. Mohammedi, Phys.Lett. **B268**, 12 (1991)

[9] N. Mohammedi, *Constant curvature and non-perturbative W3 gravity*, arXiv:hep-th/9109031

[10] T.T. Burwick and A.H. Camseddine, Nucl.Phys. **B384**, 411 (1992), arXiv:hep-th/9204002

[11] A.H. Camseddine, Nucl.Phys. **B368**, 98 (1992)

[12] I.L. Shapiro, *Perturbative approach to the two-dimensional quantum gravity*, arXiv:hep-th/9509065

[13] O. Andreev, Phys.Rev. **D57**, 3725 (1998), arXiv:hep-th/9710107

[14] J. Polchinski, *String Theory, An introduction to bosonic string theory*, Vol. 1, Cambridge University Press (1998)

[15] Distler and Kawai, Nucl.Phys. **B321,** 509 (1989)

[16] K. Becker, *Strings, Black Holes and Conformal Field Theory*, PhD Thesis. arXiv:hep-th/9404157

[17] E. Witten, Phys.Rev. **D44**, 314 (1991)

[18] M. Goulian and M. Li, Phys.Rev.Lett. **66**, 2051 (1991)

[19] P. Di Francesco and D. Kutasov, Phys.Lett. **B261**, 385 (1991). P. Di Francesco and D. Kutasov, Nucl.Phys. **B375** 119 (1992)

[20] V.S. Dotsenko and V.A. Fateev, Nucl.Phys. **B251**, 691 (1985)

[21] V.S. Dotsenko, Mod.Phys.Lett. **A6**, 3601 (1991)

[22] I.I. Kogan and A. Lewis, Nucl.Phys. **B509,** 687 (1998). arXiv:hep-th/9705240